

# COMPLEX GRADIENT SYSTEMS

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**ABSTRACT.** Let  $\tilde{M}$  be a complex manifold of complex dimension  $n+k$ . We say that the functions  $u_1, \dots, u_k$  and the vector fields  $\xi_1, \dots, \xi_k$  on  $\tilde{M}$  form a *complex gradient system* if  $\xi_1, \dots, \xi_k, J\xi_1, \dots, J\xi_k$  are linearly independent at each point  $p \in \tilde{M}$  and generate an integrable distribution of  $T\tilde{M}$  of dimension  $2k$  and  $du_\alpha(\xi_\beta) = 0$ ,  $d^c u_\alpha(\xi_\beta) = \delta_{\alpha\beta}$  for  $\alpha, \beta = 1, \dots, k$ .

We prove a Cauchy theorem for such complex gradient systems with initial data along a CR-submanifold of type  $(n, k)$ .

We also give a complete local characterization for the complex gradient systems which are *holomorphic* and *abelian*, which means that the vector fields  $\xi_\alpha^c = \xi_\alpha - iJ\xi_\alpha$ ,  $\alpha = 1, \dots, k$  are holomorphic and satisfy  $[\xi_\alpha^c, \xi_\beta^c] = 0$  for each  $\alpha, \beta = 1, \dots, k$ .

## 1. INTRODUCTION

Let  $\tilde{M}$  be a complex manifold,  $T\tilde{M}$  its (real) tangent space endowed by its complex structure  $J$ .

In [7] the authors introduced a geometric tool named *one dimensional calibrated foliation* on the complex manifold  $\tilde{M}$ . It consists of a real function  $u: \tilde{M} \rightarrow \mathbb{R}$  and a vector field  $\xi \in \Gamma(\tilde{M}, T\tilde{M})$  which satisfy the conditions

$$\begin{aligned} [\xi, J\xi] &= 0, \\ du(\xi) &= 0, \\ d^c u(\xi) &= 1. \end{aligned}$$

Here  $[\cdot, \cdot]$  is the Poisson Lie bracket between vector fields and  $d^c u(\xi) = -du(J(\xi))$ .

Among the results proved in [7] there is a Cauchy-like theorem for one dimensional calibrated foliation (see Theorem 3.1) which states the following: if  $M \subset \tilde{M}$  is a real hypersurface and  $\xi_0$  is a vector field on  $M$  which is transversal to the holomorphic tangent space to  $M$ , then, under the assumption that the integral curves  $t \mapsto \gamma(t)$  of  $\xi_0$  are real analytic, there exists a (unique) one dimensional calibrated foliation  $(\xi, u)$ , defined in a suitable neighbourhood of  $M$  in  $\tilde{M}$ , such that the vector field  $\xi$  extends  $\xi_0$ .

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The notion one dimensional calibrated foliation was motivated by the problem of finding for  $M$  an equation satisfying the homogeneous complex Monge-Ampère equation. The method has subsequently been applied in [9], [8] to prove the existence of adapted complex structures on the symplectization of a pseudo-Hermitian manifolds. The key point in proving the existence of a calibrated foliation is the construction of a function  $\tilde{G}(z, p) : D \rightarrow \tilde{M}$ , where  $D \subset M \times \mathbb{C}$  is an open neighbourhood of  $M \times \{0\}$ , which is holomorphic in  $z$  for each  $p \in M$  and for  $z = t$  real the map  $t \mapsto \tilde{G}(t, p)$  is the integral curve of the vector field  $\xi_0$  such that  $\tilde{G}(0, p) = p$ . This idea goes back to Duchamp and Kalka (see [3]).

The purpose of this paper is to provide a natural higher dimensional generalization of the notion of one dimensional calibrated foliations.

Let  $\tilde{M}$  be a complex manifold of complex dimension  $n + k$ . We say that the functions  $u_1, \dots, u_k$  and the vector fields  $\xi_1, \dots, \xi_k$  on  $\tilde{M}$  form a *complex gradient system* (of dimension  $k$ ) if

$$\xi_1, \dots, \xi_k, J(\xi_1), \dots, J(\xi_k)$$

are linearly independent at each point  $p \in \tilde{M}$  and generate an integrable distribution of  $T\tilde{M}$  of dimension  $2k$  and

$$du_\alpha(\xi_\beta) = 0, \quad d^c u_\alpha(\xi_\beta) = \delta_{\alpha\beta}$$

for  $\alpha, \beta = 1, \dots, k$ . Here  $\delta_{\alpha\beta}$  is the usual Kronecker symbol.

In a more intrinsic way a complex gradient system is given by a real vector space  $\mathcal{V}$  of dimension  $k$ , a linear monomorphism  $\rho : \mathcal{V} \rightarrow \Gamma(\tilde{M}, T\tilde{M})$ , the *representation map*, and a map  $U : \tilde{M} \rightarrow \mathcal{V}$ , the *gradient map*, which satisfy

$$\begin{aligned} dU(\rho(V)) &= 0, \\ d^c U(\rho(V)) &= V \end{aligned}$$

for each  $V \in \mathcal{V}$ .

The name “complex gradient system” (instead of “calibrated foliation”) arises from the fact that there are examples of triples  $(\mathcal{V}, \rho, U)$  where  $\mathcal{V} = \mathfrak{g}$  is the Lie algebra of a Lie group  $G$  which is a compact real form of a reductive complex Lie group  $G^\mathbb{C}$  and the map  $U$  (with the identification of  $\mathfrak{g} = \mathfrak{g}^*$  with its dual  $\mathfrak{g}^*$  by the Killing form) is the moment map associated to a symplectic action of  $G$ . It is customary in the symplectic geometry to call such a moment map as “gradient map” (see e.g. [4]) and hence the name “complex gradient system”.

For a general reference on symplectic geometry and moment maps theory see e.g. [6] and [1]. Basic definitions and notions in CR-geometry can be found in [2].

Let now describe in more details the content of the paper.

In Section 2 contains the elementary properties of a complex gradient system. In particular, any complex gradient system satisfies the formal commutativity property

$$[\rho^{\mathbb{C}}(V), \rho^{\mathbb{C}}(W)] = 0$$

for each  $V, W \in \mathcal{V}$ , where  $\rho^{\mathbb{C}}(V) = \frac{1}{2}(\rho(V) - iJ\rho(V))$  (and similarly  $\rho^{\mathbb{C}}(W) = \frac{1}{2}(\rho(W) - iJ\rho(W))$ ) is the complex vector field of type  $(1, 0)$  naturally associated to the real vector field  $\rho(V)$  (resp.  $\rho(W)$ ). See Theorem 2.1.

In section 3 we solve a Cauchy problem for a complex gradient system on a complex manifold  $\tilde{M}$  of dimension  $n+k$  with initial data on a CR-submanifold of  $\tilde{M}$  of type  $(n, k)$  (Theorem 3.1).

In section 4 we give a couple of examples applying our construction to the case of the complexification of a real Lie group  $G$ . In particular, we find explicitly the complex gradient system associated to the standard representation of the Lie algebra  $\mathfrak{g}$  of  $G$  as left invariant vector fields on  $G$ .

Finally, in the last section we give a complete local description of any *abelian holomorphic complex gradient system*  $(\mathcal{V}, \rho, U)$ , where abelian means that  $[\rho^{\mathbb{C}}(V), \overline{\rho^{\mathbb{C}}(W)}] = 0$  for each pair of vectors  $V, W \in \mathcal{V}$  and holomorphic means that  $\rho^{\mathbb{C}}(V)$  is a holomorphic vector field on  $\tilde{M}$  for each  $V \in \mathcal{V}$ . See Theorem 5.1 for details.

## 2. COMPLEX GRADIENT SYSTEMS

Let  $\tilde{M}$  be a complex manifold of complex dimension  $n+k$ ,  $T\tilde{M}$  its (real) tangent space endowed by its complex structure  $J$ . Let  $\mathcal{V}$  be a real vector space and let  $\rho : \mathcal{V} \rightarrow \Gamma(\tilde{M}, T\tilde{M})$  be a linear map. We denote by  $\mathcal{D}_{\rho}^{\mathbb{R}} \subset T\tilde{M}$  the distribution generated by the vector fields of the form  $\rho(V)$ ,  $V \in \mathcal{V}$ . We also denote by  $\mathcal{D}_{\rho}^{\mathbb{C}} \subset T\tilde{M}$  the distribution

$$\mathcal{D}_{\rho}^{\mathbb{C}} = \mathcal{D}_{\rho}^{\mathbb{R}} + J(\mathcal{D}_{\rho}^{\mathbb{R}})$$

generated by the vector fields of the form  $\rho(V)$  and  $J(\rho(V))$ ,  $V \in \mathcal{V}$ .

**Definition 2.1.** A complex gradient system of dimension  $k$  on  $\tilde{M}$  is a triple

$$(\mathcal{V}, \rho, U)$$

where:

- (1)  $\mathcal{V}$  is a real vector space of dimension  $k$ ;
- (2)  $\rho : \mathcal{V} \rightarrow \Gamma(\tilde{M}, T\tilde{M})$  is a  $\mathbb{R}$ -linear map;
- (3)  $U : \tilde{M} \rightarrow \mathcal{V}$  is a smooth map.

which satisfies

- i) for each  $V \in \mathcal{V}$  the vector field  $\rho(V)$  is smooth and we have the identities

$$\begin{aligned} dU(\rho(V)) &= 0, \\ d^c U(\rho(V)) &= V. \end{aligned}$$

ii) the distribution  $\mathcal{D}_\rho^\mathbb{C} \subset T\tilde{M}$  is integrable.

The maps  $\rho$  and  $U$  are said respectively the *representation* and the *gradient map* of the complex gradient system  $(\mathcal{V}, \rho, U)$ .

If  $\{V_1, \dots, V_k\}$  is a basis of  $\mathcal{V}$  we set

$$\xi_1 = \rho(V_1), \dots, \xi_k = \rho(V_k)$$

and for some smooth functions  $u_1, \dots, u_k$  we have

$$U = u_1 V_1 + \dots + u_k V_k.$$

Then  $(\mathcal{V}, \rho, U)$  is a complex gradient system if, and only if,

$$\begin{aligned} du_\alpha(\xi_\beta) &= 0, \quad \alpha, \beta = 1, \dots, k, \\ d^c u_\alpha(\xi_\beta) &= \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, k, \end{aligned}$$

and  $\{\xi_1, J\xi_1, \dots, \xi_k, J\xi_k\}$  is a basis of an integrable distribution.

**Proposition 2.1.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system on the manifold  $\tilde{M}$ . Then*

$$\begin{aligned} \mathcal{D}_\rho^\mathbb{R} &= \mathcal{D}_\rho^\mathbb{C} \cap \ker dU, \\ \mathcal{D}_\rho^\mathbb{C} &= \mathcal{D}_\rho^\mathbb{R} \oplus J\mathcal{D}_\rho^\mathbb{R}, \\ T\tilde{M} &= J\mathcal{D}_\rho^\mathbb{R} \oplus \ker dU. \end{aligned}$$

*Proof.* Let  $p \in \tilde{M}$  and  $v \in T_p\tilde{M}$ . Assume that  $v \in \mathcal{D}_\rho^\mathbb{C}$ . Then there are  $V, W \in \mathcal{V}$  such that  $v = \rho(V)_p + J\rho(W)_p$ . It follows

$$dU(v) = dU(\rho(V)_p) + dU(J\rho(W)_p) = -W,$$

whence

$$v \in \ker dU \iff W = 0 \iff v = \rho(V)_p \in \mathcal{D}_\rho^\mathbb{R}.$$

This proves the first assertion of the proposition.

As for the second one it suffices to prove that  $\mathcal{D}_\rho^\mathbb{R} \cap J\mathcal{D}_\rho^\mathbb{R} = 0$ . Let  $v \in \mathcal{D}_\rho^\mathbb{R} \cap J\mathcal{D}_\rho^\mathbb{R}$ . Then  $v = \rho(V)_p = J\rho(W)_p$  for some  $V, W \in \mathcal{V}$ . We then have

$$0 = dU(\rho(V)_p) = dU(J\rho(W)_p) = -W$$

and hence  $v = J\rho(W)_p = 0$ , as required.

Let now  $v \in T_p\tilde{M}$  be arbitrary and set  $V = dU(v)$ ,  $w = \rho(V)_p$ . Clearly,  $v = (v - Jw) + Jw$ . Observe that

$$dU(v - Jw) = dU(v) - dU(Jw) = V - dU(J\rho(V)_p) = V - V = 0,$$

i.e.  $v - Jw \in \ker dU$  and  $Jw \in J\mathcal{D}_\rho^\mathbb{R}$ .

If  $v \in J\mathcal{D}_\rho^\mathbb{R} \cap \ker dU$ , then  $v = J\rho(W)_p$  for some  $W \in \mathcal{V}$ . It follows that

$$0 = dU(v) = dU(J\rho(W)_p) = -W$$

and hence  $v = J\rho(W)_p = J\rho(0)_p = 0$ . This proves the last assertion of the proposition.

□

**Definition 2.2.** Given a complex gradient system  $(\mathcal{V}, \rho, U)$  we denote  $\mathcal{H}_\rho$  the distribution  $\ker dU \cap \ker d^c U$ .

**Proposition 2.2.** Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system on the manifold  $\tilde{M}$ . Then

$$\begin{aligned}\ker dU &= \mathcal{D}_\rho^\mathbb{R} \oplus \mathcal{H}_\rho, \\ T\tilde{M} &= \mathcal{D}_\rho^\mathbb{R} \oplus J\mathcal{D}_\rho^\mathbb{R} \oplus \mathcal{H}_\rho.\end{aligned}$$

*Proof.* The second equality easily follows from the first in view of the equality  $T\tilde{M} = J\mathcal{D}_\rho^\mathbb{R} \oplus \ker dU$  proved in the last proposition. So it suffices to prove that  $\ker dU = \mathcal{D}_\rho^\mathbb{R} \oplus \mathcal{H}_\rho$ .

By definition we have  $\mathcal{D}_\rho^\mathbb{R} \subset \ker dU$  and by construction  $\mathcal{H}_\rho \subset \ker dU$  so that  $\mathcal{D}_\rho^\mathbb{R} \oplus \mathcal{H}_\rho \subset \ker dU$ .

Let  $v \in \ker dU$  be arbitrary. Set  $V = d^c U(v)$  and  $w = \rho(V)_p$ . Then we have immediately  $v = (v - w) + w$  and  $w, v - w \in \ker dU$ . Observe that

$$d^c U(v - w) = d^c U(v) - d^c U(w) = V - d^c U(\rho(V)_p) = V - V = 0,$$

that is  $v - w \in \ker d^c U$  and  $w \in \mathcal{D}_\rho^\mathbb{R}$ .

Assume now that  $v \in \mathcal{D}_\rho^\mathbb{R} \cap \ker d^c U$ . Then  $v = \rho(W)_p$  for some  $W \in \mathcal{V}$ . It follows that

$$0 = d^c U(v) = d^c U(\rho(W)_p) = W$$

and hence  $v = \rho(W)_p = \rho(0)_p = 0$ . The proof is now complete.

□

**Proposition 2.3.** Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system on the manifold  $\tilde{M}$  and  $V, W \in \mathcal{V}$ . Then

$$[\rho(V), \rho(W)], [\rho(V), J\rho(W)], [J\rho(V), J\rho(W)] \in \Gamma(\tilde{M}, \mathcal{D}_\rho^\mathbb{R}).$$

Moreover

$$\begin{aligned}\mathrm{dd}^c U(\rho(V), \rho(W)) &= -d^c U([\rho(V), \rho(W)]) \\ \mathrm{dd}^c U(J\rho(V), J\rho(W)) &= -d^c U([\rho(V), \rho(W)]) \\ \mathrm{dd}^c U(\rho(V), J\rho(W)) &= -d^c U([\rho(V), J\rho(W)]).\end{aligned}$$

*Proof.* Let  $\xi_1$  be either  $\rho(V)$  or  $J\rho(V)$  and  $\xi_2$  be either  $\rho(W)$  or  $J\rho(W)$ . Then  $\xi_1(U)$  and  $\xi_2(U)$  are constant functions and hence

$$\xi_2(\xi_1(U)) = \xi_1(\xi_2(U)) = 0.$$

This easily implies that  $[\xi_1, \xi_2] \in \Gamma(\tilde{M}, \ker dU)$ . By definition of complex gradient system,  $\mathcal{D}_\rho^\mathbb{C}$  is an integrable distribution, so, in view of the last proposition, we have  $\mathcal{D}_\rho^\mathbb{R} = \mathcal{D}_\rho^\mathbb{C} \cap \ker dU$  and hence  $[\xi_1, \xi_2] \in \Gamma(\tilde{M}, \mathcal{D}_\rho^\mathbb{R})$ .

Using again the equalities  $\xi_2(\xi_1(U)) = \xi_1(\xi_2(U)) = 0$  we also obtain

$$\begin{aligned}\mathrm{dd}^c U(\xi_1, \xi_2) &= \xi_1(\xi_2(U)) - \xi_2(\xi_1(U)) - \mathrm{dd}^c U([\xi_1, \xi_2]) \\ &= -\mathrm{dd}^c U([\xi_1, \xi_2]).\end{aligned}$$

This completes the proof of the proposition.

□

**Corollary 2.1.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system on the manifold  $\tilde{M}$  and  $V, W \in \mathcal{V}$ . Then*

$$\begin{aligned}\rho(\mathrm{dd}^c U(\rho(V), \rho(W))) &= -[\rho(V), \rho(W)] \\ \rho(\mathrm{dd}^c U(J\rho(V), J\rho(W))) &= -[\rho(V), \rho(W)] \\ \rho(\mathrm{dd}^c U(\rho(V), J\rho(W))) &= -[\rho(V), J\rho(W)].\end{aligned}$$

*Proof.* Apply  $\rho$  to both sides of the last three equalities of the previous proposition and use the identity  $\mathrm{d}^c U(\rho(V)) = V$ .

□

**Corollary 2.2.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system on the manifold  $\tilde{M}$ . Then the distribution  $\mathcal{D}_\rho^\mathbb{R} \subset T\tilde{M}$  is integrable.*

**Corollary 2.3.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system of dimension  $k$  on the complex manifold  $\tilde{M}$  of dimension  $n+k$ . For every  $V \in \mathcal{V}$  the level set  $U^{-1}(V)$  of the smooth function  $U$  is either empty or it is a CR-submanifold of type  $(n, k)$ .*

**Proposition 2.4.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system on the manifold  $\tilde{M}$ . For every  $V, W \in \mathcal{V}$  one has*

$$\begin{aligned}[J\rho(V), J\rho(W)] &= [\rho(V), \rho(W)], \\ [J\rho(V), \rho(W)] &= -[\rho(V), J\rho(W)].\end{aligned}$$

*Proof.* Since  $\tilde{M}$  is a complex manifold the complex structure  $J$  is integrable, hence

$$J[\rho(V), \rho(W)] - J[J\rho(V), J\rho(W)] = [J\rho(V), \rho(W)] + [\rho(V), J\rho(W)].$$

The right side of such equality belongs to  $\Gamma(\tilde{M}, J\mathcal{D}_\rho^\mathbb{R})$  while the second belongs to  $\Gamma(\tilde{M}, \mathcal{D}_\rho^\mathbb{R})$ . Since  $J\mathcal{D}_\rho^\mathbb{R} \cap \mathcal{D}_\rho^\mathbb{R} = 0$  it follows that

$$\begin{aligned}J[\rho(V), \rho(W)] - J[J\rho(V), J\rho(W)] &= 0, \\ [J\rho(V), \rho(W)] + [\rho(V), J\rho(W)] &= 0\end{aligned}$$

and the assertion follows.

□

Let  $T_\mathbb{C}\tilde{M} = \mathbb{C} \otimes_\mathbb{R} T\tilde{M}$  denote the complexification of  $T\tilde{M}$  and  $T_\mathbb{C}^{(1,0)}\tilde{M}$  the subbundle of the tangent vector of type  $(1, 0)$ .

**Definition 2.3.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system. The complexified representation*

$$\rho^\mathbb{C} : \mathcal{V} \rightarrow \Gamma(\tilde{M}, T_\mathbb{C}^{(1,0)}\tilde{M})$$

*is defined for each  $V \in \mathcal{V}$  by*

$$\rho^\mathbb{C}(V) = \frac{1}{2}(\rho(V) - iJ(\rho(V)))$$

With a little abuse of language we say that  $\rho^{\mathbb{C}}$  is holomorphic if  $\rho^{\mathbb{C}}(V)$  is a holomorphic vector field on  $\tilde{M}$  for each  $V \in \mathcal{V}$ .

With this notation the last proposition can be restated as follows.

**Theorem 2.1.** *Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system. Then for each  $V, W \in \mathcal{V}$  we have*

$$[\rho^{\mathbb{C}}(V), \rho^{\mathbb{C}}(W)] = 0.$$

### 3. A CAUCHY PROBLEM

Let  $\tilde{M}$  be a complex manifold of complex dimension  $n+k$ . Let  $M \subset \tilde{M}$  be a CR-submanifold of  $\tilde{M}$  of type  $(n, k)$ .

**Definition 3.1.** *Let  $\mathcal{V}$  be a real vector space. A linear map  $\rho_0 : \mathcal{V} \rightarrow \Gamma(M, TM)$  is said to be CR-transverse if for each  $V \in \mathcal{V} \setminus \{0\}$  and each  $p \in M$  we have  $J(\rho_0(V)(p)) \notin T_p M$ .*

Given a vector field  $X \in \Gamma(M, TM)$  we denote by  $\text{Exp}_p(X)$  the exponential mapping associated to the vector field  $X$ : if  $\gamma(t)$  is the integral curve of the vector field  $X$  such that  $\gamma(0) = p$  then  $\text{Exp}_p(X) = \gamma(1)$ .

Let  $\rho_0 : \mathcal{V} \rightarrow \Gamma(M, TM)$  be a linear map of real vector spaces. The flow associated to  $\rho_0$  is defined for  $p \in M$  and  $V \in \mathcal{V}$  by

$$G_{\rho_0}(p, V) = \text{Exp}_p(\rho_0(V))$$

$G_{\rho_0}$  is a smooth map which is well defined in an open neighbourhood of  $M \times \{0\}$  in  $M \times \mathcal{V}$ .

Let  $\mathcal{V}^{\mathbb{C}}$  denote the complexification  $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{V}$  of the real vector space  $\mathcal{V}$ . We then say that the flow  $G_{\rho_0}$  is *uniformly (real) analytic* if there exist an open neighbourhood  $D \subset M \times \mathcal{V}^{\mathbb{C}}$  of  $M \times \{0\}$  and a smooth function

$$\tilde{G}_{\rho_0} : D \rightarrow M$$

which coincides with  $G_{\rho_0}$  on  $M \times \mathcal{V}$  and for each  $p \in M$  the map defined on the open set

$$D_p = \{V \in \mathcal{V}^{\mathbb{C}} \mid (p, V) \in D\}$$

by

$$V \mapsto \tilde{G}_{\rho_0}(p, V)$$

is holomorphic.

The map  $\tilde{G}_{\rho_0}$  will be called the *complexification* of the flow  $G_{\rho_0}$ .

Let  $F_{\rho_0}$  denote the restriction of  $\tilde{G}_{\rho_0}$  to  $\tilde{D} = D \cap M \times i\mathcal{V}$ . As it is immediately seen, shrinking the domain  $D$  if necessary, the map  $F_{\rho_0}$  is a diffeomorphism between  $\tilde{D}$  and  $F_{\rho_0}(\tilde{D})$  if, and only if, the map  $\rho_0$  is CR-transverse.

In this case we denote by  $U_{\rho_0} : F_{\rho_0}(\tilde{D}) \rightarrow \mathcal{V}$  the unique map satisfying

$$U_{\rho_0}(F_{\rho_0}(p, iV)) = V.$$

for each  $p \in M$  and each  $V \in \mathcal{V}$ .

Observe that the map  $U_{\rho_0}$  vanishes exactly on  $M$ , so we will refer to it as to the *equation* of  $M$  associated to  $\rho_0$ .

We say that a complex gradient system  $(\mathcal{V}, \rho, U)$  *extends*  $\rho_0$  if it is defined in an open neighbourhood  $N \subset \tilde{M}$  of  $M$  and for every  $p \in M$  and  $V \in \mathcal{V}$

$$\rho(V)(p) = \rho_0(V)(p).$$

If  $(\mathcal{V}, \rho_1, U)$ ,  $(\mathcal{V}, \rho_2, U)$  are two extensions of  $\rho_0$  such that for every  $V \in \mathcal{V}$  the sections  $\rho_1(V)$ ,  $\rho_2(V)$  coincide on  $N$  then we write  $\rho_1|_N = \rho_2|_N$ .

**Theorem 3.1.** *Let  $\tilde{M}$  be a complex manifold of complex dimension  $n + k$ ,  $M \subset \tilde{M}$  a CR-submanifold of  $\tilde{M}$  of type  $(n, k)$ . Let  $\mathcal{V}$  be a real vector space and  $\rho_0 : \mathcal{V} \rightarrow \Gamma(M, TM)$  a CR-transverse linear map such that the distribution  $\mathcal{D}_{\rho_0}^{\mathbb{R}}$  is integrable. Assume that the associated flow  $G_{\rho_0}$  is uniformly real analytic and let  $U_{\rho_0}$  be the associated equation.*

*Then there exists an open neighbourhood  $N \subset \tilde{M}$  of  $M$  and an  $\mathbb{R}$ -linear map*

$$\rho : \mathcal{V} \rightarrow \Gamma(N, T\tilde{M})$$

*such that  $(\mathcal{V}, \rho, U_{\rho_0})$  is a complex gradient system which extends  $\rho_0$ .*

*The map  $\rho$  is unique in a neighbourhood of  $M$ , that is if*

$$\rho_1 : \mathcal{V} \rightarrow \Gamma(N_1, T\tilde{M}), \quad \rho_2 : \mathcal{V} \rightarrow \Gamma(N_2, T\tilde{M})$$

*are  $\mathbb{R}$ -linear maps such that  $(\mathcal{V}, \rho_1, U_{\rho_0})$  and  $(\mathcal{V}, \rho_2, U_{\rho_0})$  are complex gradient systems which extend  $\rho_0$  then  $\rho_1|_N = \rho_2|_N$  for a suitable open neighbourhood  $N \subset N_1 \cap N_2$  of  $M$ .*

*Proof.* It is not restrictive to assume that the map  $\tilde{G}_{\rho_0}$  is a diffeomorphism between  $\tilde{D} = D \cap M \times i\mathcal{V}$  and  $F_{\rho_0}(\tilde{D})$ .

Fix a basis  $\{V_1, \dots, V_k\}$  of  $\mathcal{V}$  and set

$$\xi_\alpha^0 = \rho(V_\alpha), \quad \alpha = 1, \dots, k.$$

Let  $J$  denote the complex structure on  $T\tilde{D}$  induced by the pullback of of the complex structure on  $TF_{\rho_0}(\tilde{D}) \subset T\tilde{M}$  under the map  $F_{\rho_0}$ .

It is not restrictive to identify the neighbourhood  $F_{\rho_0}(\tilde{D})$  of  $M$  in  $\tilde{M}$  with the domain  $\tilde{D} \subset M \times i\mathcal{V}$ . We also identify  $M \times i\mathcal{V}$  with  $M \times \mathbb{R}^k$  by

$$M \times \mathbb{R}^k \ni (p, u_1, \dots, u_k) \mapsto (p, iu_1V_1 + \dots + iu_kV_k) \in M \times i\mathcal{V}.$$

Let

$$U = (u_1, \dots, u_k) : M \times \mathbb{R}^k \rightarrow \mathbb{R}^k \simeq \mathcal{V}$$

be the projection on the second factor.

We will prove the existence of the required complex gradient system showing that there exist vector fields

$$\xi_\alpha \quad \alpha = 1, \dots, k$$

defined in a suitable neighbourhood  $N$  of  $M \times \{0\}$  in  $\tilde{D}$  such that

$$du_\alpha(\xi_\beta) = 0 \quad \alpha, \beta = 1, \dots, k$$



and

$$d^c u_\alpha(\xi_\beta) = \delta_{\alpha\beta} \quad \alpha, \beta = 1, \dots, k.$$

Let  $\tilde{\xi}_\alpha^0$ ,  $\alpha = 1, \dots, k$ , denote the vector fields on  $\tilde{D}$  which coincide with  $\xi_\alpha^0$  on  $M \times \{0\}$  and are invariant under the action  $\mathcal{V} \times (M \times i\mathcal{V})$  given by

$$(W, (p, V)) \mapsto (p, W + V).$$

Let also  $\mathcal{D}$  denote the distribution on  $T(M \times i\mathcal{V}) \approx T(M \times \mathbb{R}^k)$  generated by the vector fields

$$\tilde{\xi}_1^0, \dots, \tilde{\xi}_k^0, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_k}.$$

$\mathcal{D}$  is completely integrable and the maximal integral submanifolds of  $\mathcal{D}$  are of the form  $S \times \mathbb{R}^k$  where  $S$  is a maximal integral submanifold of the distribution  $\mathcal{D}_\rho^\mathbb{R}$ .

By construction, the intersection of each maximal integral submanifolds of  $\mathcal{D}$  with the domain  $\tilde{D}$  is a complex submanifold of  $\tilde{D}$  of complex dimension  $k$ . Moreover, for each  $p \in M$  and each  $\alpha = 1, \dots, k$  we have

$$J(\tilde{\xi}_\alpha^0)(p) = \frac{\partial}{\partial u_\alpha}(p).$$

Let  $P = (p_{\alpha\beta})$ ,  $Q = (q_{\alpha\beta})$  be the square matrices of order  $k$  with entry smooth function on  $\tilde{D}$  defined by

$$\begin{aligned} p_{\alpha\beta} &= J(\tilde{\xi}_\beta^0)(u_\alpha), \\ q_{\alpha\beta} &= J\left(\frac{\partial}{\partial u_\beta}\right)(u_\alpha). \end{aligned}$$

Observe that for each  $p \in M$  the matrices  $P((p, 0))$  and  $Q((p, 0))$  are respectively the identity matrix and the zero matrix of order  $k$ .

Let  $N$  be the open neighbourhood of  $M \times \{0\}$  in  $\tilde{D}$  defined by

$$N = \left\{ (p, u_1, \dots, u_k) \in \tilde{D} \mid \det P((p, u_1, \dots, u_k)) \neq 0 \right\}.$$

Denote  $A = (a_{\alpha\beta})$  the matrix  $P^{-1}Q$ , and set

$$\xi_\alpha = -J\left(\frac{\partial}{\partial u_\alpha}\right) + \sum_{\beta=1}^k a_{\beta\alpha} J(\tilde{\xi}_\beta^0) \quad \alpha = 1, \dots, k.$$

Then

$$J(\xi_\alpha) = \frac{\partial}{\partial u_\alpha} - \sum_{\beta=1}^k a_{\beta\alpha} \tilde{\xi}_\beta^0 \quad \alpha = 1, \dots, k$$

and, in view of the  $J$ -invariance of the distribution  $\mathcal{D}$  it follows that

$$\xi_1, \dots, \xi_k, J(\xi_1), \dots, J(\xi_k)$$

generate the distribution  $\mathcal{D}$  on  $N$ . It is easy to check that

$$du_\alpha(\xi_\beta) = 0 \quad \alpha, \beta = 1, \dots, k$$

and

$$d^c u_\alpha(\xi_\beta) = \delta_{\alpha\beta} \quad \alpha, \beta = 1, \dots, k,$$

as required.

In order to prove the uniqueness of the map  $\rho$  assume that the complex gradient systems  $(\mathcal{V}, \rho_1, U_{\rho_0})$ ,  $(\mathcal{V}, \rho_2, U_{\rho_0})$  extend  $\rho_0$  and set  $\gamma = \rho_1 - \rho_2$ .

We are going to prove that, after shrinking  $N$  if necessary,  $\gamma_N = 0$  showing before that the complex distributions  $\mathcal{D}_{\rho_1}^{\mathbb{C}}$  and  $\mathcal{D}_{\rho_2}^{\mathbb{C}}$  associated respectively to  $\rho_1$  and  $\rho_2$  coincide near to  $M \times \{0\}$ .

By hypothesis the distribution  $\mathcal{D}_{\rho_0}^{\mathbb{R}}$  is integrable and its maximal integral submanifold are real submanifolds of  $M$  of (real) dimension  $k$ . For every  $p \in M$  consider the maximal integral submanifolds  $S_1, S_2$  through  $p$  of  $\mathcal{D}_{\rho_1}^{\mathbb{C}}$  and  $\mathcal{D}_{\rho_2}^{\mathbb{C}}$  respectively. Since  $\rho_1$  and  $\rho_2$  both extend  $\rho_0$  it follows that

$$S^{\mathbb{R}} = S_1 \cap M = S_2 \cap M$$

is the maximal integral (real) submanifold of (real) dimension  $k$  of the distribution  $\mathcal{D}_{\rho_0}^{\mathbb{R}}$  through  $p$ . In view of the hypothesis of CR-transversality,  $S^{\mathbb{R}}$  is a totally real submanifold of  $S_1$  and  $S_2$ . It follows that  $S_1 = S_2$ .

We have so proved that the maximal integral submanifolds of the distributions  $\mathcal{D}_{\rho_1}^{\mathbb{C}}$  and  $\mathcal{D}_{\rho_2}^{\mathbb{C}}$  which meet the submanifold  $M$  are the same and consequently, after shrinking  $N$  if necessarily, it follows that the distributions  $\mathcal{D}_{\rho_1}^{\mathbb{C}}$  and  $\mathcal{D}_{\rho_2}^{\mathbb{C}}$  coincide on  $N$ .

Let now  $V \in \mathcal{V}$  be an arbitrary vector. Then,  $\gamma(V) \in \Gamma(N, \mathcal{H}_{\rho_1})$  and the above argument shows that  $\gamma(V) \in \Gamma(N, \mathcal{D}_{\rho_1}^{\mathbb{C}})$ . Since  $T\tilde{M} = \mathcal{D}_{\rho_1}^{\mathbb{R}} \oplus J\mathcal{D}_{\rho_1}^{\mathbb{R}} \oplus \mathcal{H}_{\rho_1}$  it follows that  $\gamma(V)|_N = 0$  and this ends the proof  $V \in \mathcal{V}$  being arbitrary.  $\square$

When  $k = \dim \mathcal{V} = 1$  the result above is contained in [7, Theorem 3.1] where a stronger uniqueness result was obtained. Namely, if  $(\xi_1, u_1)$  and  $(\xi_2, u_2)$  are two one dimensional calibrated foliation such that  $\xi_1$  and  $\xi_2$  both extend  $\xi_0$  along the hypersurface  $M$  then  $\xi_1 = \xi_2$  and  $u_1 = u_2$  in a neighbourhood of  $M$ .

Such a uniqueness result does not hold for a general complex gradient system. Indeed, consider  $\tilde{M} = \mathbb{C}$ ,  $M = \mathbb{R}$  and

$$\xi_0 = \frac{\partial}{\partial x}.$$

Then our construction yields the gradient map

$$U(z) = U(x + iy) = -\operatorname{Re}(z) = -x$$

and the vector field

$$\xi = \frac{\partial}{\partial x},$$

but also the pair  $(\xi_1, U_1)$  where

$$U_1(z) = U_1(x + iy) = e^{-y} - 1$$

and

$$\xi_1 = e^y \frac{\partial}{\partial x},$$

is a complex gradient system which extends  $\xi_0$ .

In the case  $k = 1$  the condition  $[\xi, J\xi] = 0$  which is in the definition of one dimensional calibrated foliation given in [7] ensures the uniqueness for the Cauchy problem (see [7, Theorem 3.1]). It is not clear which is the right condition (if any) to add in order to guarantee the uniqueness also in this non commutative setting.

See also the examples given in the next section.

#### 4. LIE GROUPS

Let  $G^{\mathbb{C}}$  be a complex Lie group of (complex) dimension  $k$  which is the complexification of a real Lie group  $G$ ;  $G$  is totally real submanifold of  $G^{\mathbb{C}}$ . Let  $\mathfrak{g}^{\mathbb{C}}$  and  $\mathfrak{g}$  be the Lie algebras of  $G^{\mathbb{C}}$  and  $G$  respectively.

We identify  $\mathfrak{g}$  (resp.  $\mathfrak{g}^{\mathbb{C}}$ ) with the tangent space to  $G$  ( $G^{\mathbb{C}}$ ) at the origin and for each  $V \in \mathfrak{g}$  (resp.  $V \in \mathfrak{g}^{\mathbb{C}}$ ) we denote by  $L_V$  the corresponding left invariant vector field on  $G$  (resp.  $G^{\mathbb{C}}$ ). The complexification of the flow associated to the map  $\mathfrak{g}^{\mathbb{C}} \ni V \mapsto L_V$  is the map

$$G \times \mathfrak{g}^{\mathbb{C}} \ni (g, V) \mapsto g \exp(V) \in G^{\mathbb{C}}$$

being  $\exp$  the standard exponential map  $\exp : \mathfrak{g}^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ . Let denote by  $(\mathfrak{g}, \rho, U)$  the complex gradient system which extends  $V \mapsto L_V$ . Then we have the identity

$$U(g \exp(-iV)) = V.$$

If  $G^{\mathbb{C}}$  be a complex reductive Lie group and  $G$  is a compact real form for  $G^{\mathbb{C}}$  then we have the Cartan decomposition of  $G^{\mathbb{C}}$

$$\begin{aligned} G \times \mathfrak{g} &\rightarrow G^{\mathbb{C}} \\ (g, V) &\mapsto g \exp(iV). \end{aligned}$$

In this case the algebra  $\mathfrak{g}$  admits a definite metric  $B$ , invariant under the adjoint representation  $\text{Ad}_G$  of  $G$ , inducing an isomorphism between the Lie algebra  $\mathfrak{g}$  and its dual  $\mathfrak{g}^*$ . With this identification the gradient map  $U$  is (up to the sign) the moment map associated to a symplectic action of  $G$  on  $G^{\mathbb{C}}$ . See e.g. [5] for details.

This example explain our terminology “*complex gradient system*”.

We would like to point out that in general, as shown by the examples below, the representation  $\rho$  of the complex gradient system extending the left representation  $V \mapsto L_V$  is not the restriction of the left representation of  $\mathfrak{g}^{\mathbb{C}}$ .

Let  $G^{\mathbb{C}}$  be the matrix Lie group of the matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

with  $z_\alpha = x_\alpha + iy_\alpha \in \mathbb{C}$ ,  $\alpha = 1, 2, 3$  and let  $G$  be the corresponding group with  $z_i \in \mathbb{R}$ .

Then  $\mathfrak{g}$  is given by the matrices of the form

$$\begin{pmatrix} 0 & u_1 & u_3 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{pmatrix}$$

with  $u_\alpha \in \mathbb{R}$ ,  $\alpha = 1, \dots, 3$ .

Put

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then  $E_1, E_2, E_3$  is a basis of  $\mathfrak{g}$ . Denoting  $L_\alpha = L_{E_\alpha}$  we then have

$$\begin{aligned} L_1 &= \frac{\partial}{\partial x_1}, \\ L_2 &= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial y_3}, \\ L_3 &= \frac{\partial}{\partial x_3}, \\ JL_1 &= \frac{\partial}{\partial y_1}, \\ JL_2 &= \frac{\partial}{\partial y_2} + x_1 \frac{\partial}{\partial y_3} - y_1 \frac{\partial}{\partial x_3}, \\ JL_3 &= \frac{\partial}{\partial y_3} \end{aligned}$$

Some computation yields for the gradient map  $U$  the expression

$$U(z_1, z_2, z_3) = -y_1 E_1 - y_2 E_2 - (y_3 + x_1 y_2) E_3$$

and the representation  $\rho$  is given by

$$\begin{aligned} \rho(E_1) &= \tilde{E}_1 = \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_3} = L_1 + y_2 J(L_3), \\ \rho(E_2) &= \tilde{E}_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} = L_2 - y_1 J(L_3), \\ \rho(E_3) &= \tilde{E}_3 = \frac{\partial}{\partial x_3} = L_3. \end{aligned}$$

Observe that

$$\begin{aligned} [\tilde{E}_1, \tilde{E}_2] &= \tilde{E}_3, \quad [\tilde{E}_1, \tilde{E}_3] = [\tilde{E}_2, \tilde{E}_3] = 0, \\ [J\tilde{E}_1, J\tilde{E}_2] &= \tilde{E}_3, \quad [J\tilde{E}_1, J\tilde{E}_3] = [J\tilde{E}_2, J\tilde{E}_3] = 0, \\ [\tilde{E}_i, J\tilde{E}_j] &= 0 \quad i, j = 1, 2, 3. \end{aligned}$$

It follows that the representation  $\rho : \mathfrak{g} \rightarrow \Gamma(G^{\mathbb{C}}, TG^{\mathbb{C}})$  is a Lie algebra isomorphism and the gradient map  $U$  is a harmonic function.

Let now  $G^{\mathbb{C}}$  be the matrix Lie group of the matrices of the form

$$\begin{pmatrix} z_1 & z_2 \\ 0 & 1 \end{pmatrix}$$

with  $z_1, z_2 \in \mathbb{C}$ ,  $z_1 \neq 0$ , and let  $G$  be the corresponding group with  $z_1, z_2 \in \mathbb{R}$ .

The Lie algebra  $\mathfrak{g}$  of  $G$  is given by the matrices of the form

$$\begin{pmatrix} u_1 & u_2 \\ 0 & 0 \end{pmatrix}$$

with  $u_1, u_2 \in \mathbb{R}$ . The matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

forms a basis of the Lie algebra  $\mathfrak{g}$  which satisfies the relation

$$[E_1, E_2] = E_2.$$

The corresponding left invariant vector fields on  $G^{\mathbb{C}}$  are given by

$$\begin{aligned} L_1 &= x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1}, \\ L_2 &= x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1}, \\ JL_1 &= -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}, \\ JL_2 &= -y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2}. \end{aligned}$$

After some computations we obtain that the gradient map is given by

$$U(z_1, z_2) = -\theta_1 E_1 - \frac{y_2 \theta_1}{y_1} E_2$$

where

$$\theta_1 = \arctan \frac{y_1}{x_1}$$

and the representation  $\rho$  satisfies

$$\begin{aligned}\rho(E_1) = \tilde{E}_1 &= x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + y_2 \left( \frac{x_1}{y_1} - \frac{1}{\theta_1} \right) \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}, \\ \rho(E_2) = \tilde{E}_2 &= \frac{y_1}{\theta_1} \frac{\partial}{\partial x_2}.\end{aligned}$$

Observe that

$$[\tilde{E}_1, \tilde{E}_2] = [J\tilde{E}_1, J\tilde{E}_2] = \tilde{E}_2,$$

and

$$\begin{aligned}[\tilde{E}_1, J\tilde{E}_1] &= \frac{2y_2}{y_1} \left( \frac{x_1}{y_1} - \frac{1}{\theta_1} \right) \tilde{E}_2, \\ [\tilde{E}_1, J\tilde{E}_2] &= - \left( \frac{x_1}{y_1} - \frac{1}{\theta_1} \right) \tilde{E}_2, \\ [\tilde{E}_2, J\tilde{E}_2] &= 0,\end{aligned}$$

namely the representation  $\rho : \mathfrak{g} \rightarrow \Gamma(G^{\mathbb{C}}, TG^{\mathbb{C}})$  is a Lie algebra isomorphism but the gradient map  $U$  is not a harmonic function and the vector fields  $\tilde{E}_1, \tilde{E}_2, J\tilde{E}_1$  and  $J\tilde{E}_2$  are not a basis of a Lie sub-algebra of  $\Gamma(G^{\mathbb{C}}, TG^{\mathbb{C}})$ .

## 5. THE HOLOMORPHIC ABELIAN CASE

Let  $(\mathcal{V}, \rho, U)$  be a complex gradient system.

With a little abuse of language we say that such a complex gradient system is *holomorphic* if  $\rho^{\mathbb{C}}(V)$  is a holomorphic vector field on  $\tilde{M}$  for each  $V \in \mathcal{V}$ .

We also say that it is *abelian* if

$$[\rho^{\mathbb{C}}(V), \overline{\rho^{\mathbb{C}}(W)}] = 0$$

for each pair of vectors  $V, W \in \mathcal{V}$ . Such a condition is equivalent to say that for each pair of vectors  $V, W \in \mathcal{V}$  one has

$$[\rho(V), \rho(W)] = [\rho(V), J\rho(W)] = [J\rho(V), J\rho(W)] = 0$$

Consider now a domain  $\Omega \subset \mathbb{C}^n$  and let  $F : \Omega \rightarrow \mathbb{R}^k$  be a smooth function. We associate to  $F$  a complex gradient system as follows.

Set  $\tilde{M} = \Omega \times \mathbb{C}^k$ ,  $\mathcal{V} = \mathbb{R}^k$  and define

$$U(z, w) = F(x, y) - u,$$

where  $z = x + iy$  and  $w = t + iu$  with  $x, y \in \mathbb{R}^n$  and  $t, u \in \mathbb{R}^k$ . Finally consider the linear map  $\rho : \mathbb{R}^k \rightarrow \Gamma(\tilde{M}, T\tilde{M})$  characterized by the conditions

$$\rho(e_\alpha) = \frac{\partial}{\partial t_\alpha} \quad \alpha = 1, \dots, k,$$

where  $e_1, \dots, e_k$  is the canonical basis of  $\mathbb{R}^k$ .

It is easy to show that this complex gradient system is holomorphic and abelian and the aim of the next theorem is to prove that it is the local model

of any holomorphic abelian complex gradient system. Namely the following is true

**Theorem 5.1.** *Let  $\tilde{M}$  be a complex manifold of complex dimension  $n+k$ . Let  $(\mathbb{R}^k, \rho, U)$  be a holomorphic abelian complex gradient system on  $\tilde{M}$ . Then for each point  $p$  there exist a complex coordinate system*

$$z = (z_1, \dots, z_n), \quad w = (w_1, \dots, w_k)$$

$z_\mu = x_\mu + iy_\mu, \mu = 1, \dots, n, w_\alpha = t_\alpha + iu_\alpha, \alpha = 1, \dots, k, x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), u = (u_1, \dots, u_k)$  and a smooth (vector) function  $F$  depending only on  $x$  and  $y$  such that

$$\rho(e_\alpha) = \frac{\partial}{\partial t_\alpha} \quad \alpha = 1, \dots, k,$$

$$U(z, w) = F(x, y) - u.$$

*Proof.* Let

$$g_t^1, \dots, g_t^k, h_t^1, \dots, h_t^k$$

be the (local) one parameter group of transformation of  $\tilde{M}$  generated by the vector fields

$$\xi_1, \dots, \xi_k, J(\xi_1), \dots, J(\xi_k).$$

By hypotheses the Lie brackets between all pair of vector fields among  $\xi_1, \dots, \xi_k$  and  $J\xi_1, \dots, J\xi_k$  are zero and hence the transformations  $g_t^1, \dots, g_t^k$  and  $h_t^1, \dots, h_t^k$  commute each other.

Let  $p \in \tilde{M}$  be fixed and let  $z_1, \dots, z_{n+k}$  be a complex coordinates system around  $p$ , where  $z_\mu = x_\mu + iy_\mu, \mu = 1, \dots, n+k$ .

After reordering the coordinates we may suppose that

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}, \xi_1, J\xi_1, \dots, \xi_k, J\xi_k$$

generates the tangent space to  $\tilde{M}$  at each point in a suitable neighbourhood of  $p$ .

For  $\alpha = 1, \dots, k$  set  $w_\alpha = t_\alpha + iu_\alpha$  and define

$$G_{w_\alpha}^\alpha = g_{t_\alpha}^\alpha \circ h_{u_\alpha}^\alpha,$$

Then the map

$$(z_1, \dots, z_n, w_1, \dots, w_k) \mapsto G_{w_1}^1 \circ \dots \circ G_{w_k}^k (z_1, \dots, z_n, 0, \dots, 0),$$

is a diffeomorphism  $\varphi$  between an open set of  $\mathbb{C}^{n+k}$  and a suitable neighbourhood  $U$  of  $p$  in  $\tilde{M}$ , that is

$$x_1, y_1, \dots, x_n, y_n, t_1, u_1, \dots, t_k, u_k$$

is a real coordinate system on  $U$ .

Since the maps  $G_{w_1}^1 \dots G_{w_k}^k$  commute each other it follows that with respect to such a coordinate system we have

$$\xi_\alpha = \rho(e_\alpha) = \frac{\partial}{\partial t_\alpha}$$

for  $\alpha = 1, \dots, k$ .

We now prove

$$z_1, \dots, z_n, w_1, \dots, w_k$$

are complex coordinates on  $U$ , showing that the diffeomorphism  $\varphi$  is in fact a biholomorphism.

Since, by hypotheses,  $(\mathbb{R}^k, \rho, U)$  is a holomorphic abelian complex gradient system, it follows that  $G_{w_1}^1 \dots G_{w_k}^k$  are holomorphic local diffeomorphisms and for fixed  $w_1, \dots, w_k$  the map

$$\varphi(z_1, \dots, z_n, w_1, \dots, w_k)$$

is holomorphic with respect to the variables  $z_1, \dots, z_n$ . Moreover, for  $\alpha = 1, \dots, k$  the maps  $g_{t_\alpha}$  and  $h_{u_\alpha}$  commute and hence the map  $w_\alpha \mapsto G_{w_\alpha}^\alpha(\cdot)$  is holomorphic with respect to  $w_\alpha$ . On the other hand, since the maps  $G_{w_1}^1 \dots G_{w_k}^k$  commute each other, the map  $\varphi$  is holomorphic with respect to the variable  $w_\alpha$ ,  $\alpha = 1, \dots, k$ , when the variables  $z_1, \dots, z_n$  and  $w_1, \dots, w_{\alpha-1}, w_{\alpha+1}, \dots, w_k$  are fixed.

Thus the map  $\varphi$  is separately holomorphic in each variable and hence is holomorphic.

Finally let  $U = (U_1, \dots, U_k) : \tilde{M} \rightarrow \mathbb{R}^k$  be the gradient map and consider the map  $F = (F_1, \dots, F_k) : U \rightarrow \mathbb{R}^k$  defined by

$$F_\alpha(z, w) = F_\alpha(x, y, t, u) = U_\alpha(z, w) + u_\alpha \quad \alpha = 1, \dots, k.$$

We end the proof showing that the map  $F$  does not depend on the variables  $t$  and  $u$ . Indeed, for  $\alpha, \beta = 1, \dots, k$ , we have

$$\frac{\partial F_\alpha}{\partial t_\beta} = \xi_\beta(U_\alpha) = 0$$

and

$$\frac{\partial F_\alpha}{\partial u_\beta} = J(\xi_\beta)(U_\alpha) + \delta_{\alpha\beta} = -\delta_{\alpha\beta} + \delta_{\alpha\beta} = 0.$$

□

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